20080708 at Leiden

"Duality" between quiver varieties and double affine Grassmann $= \mathcal{G}r$

This is a duality in the quotation marks.
Not like geom. Langlands. This will become clearer later.

○ starting point

$\mathcal{G}r$ of type $SL(r)$, level n
$=$ Uhlenbeck partial cptification of moduli of $SL(r)$-b'dles on $\mathbb{C}^2/\mathbb{Z}_n$
$\subsetneqq$ quiver variety of affine type A_{n-1}, level r
(not $=$, because quiver var. $=$ $GL(r)$-b'dles)

○ <u>Bad news</u>

generalization to other types
$\mathcal{G}r$ ---- replace $SL(r)$ by other simple groups
quiver ---- replace $\mathbb{C}^2/\mathbb{Z}_{n+1}$
by other ADE singularities $\mathbb{C}^2/\Gamma$
or more general quiver via ADHM

no more intersection.
Thus this works only type A.

But $\exists$ good news indicating something interesting

- relation to representation theory

Gr of type G ---- $\widehat{G^L}$ level n

quiver of type $\mathbb{C}^2/\Gamma$ ---- $\widehat{\mathfrak{g}_\Gamma}$ level r

$G = SL(r)$, $\Gamma = \mathbb{C}^2/\mathbb{Z}_n$, the corr. rep. theories are related by <u>level-rank duality</u> (I. Frenkel)

<u>Rem.</u>

There are several generalizations, but not arbitrary G, Γ

"<u>Duality</u>"

In Gr & quiver var.

many representation theoretic informations are encoded in geometric objects (e.g. IC sheaves, cycles in homology).

geometric object in theory A

$\longleftrightarrow$ geometric obj. in theory B

s.t. for type $\hat{A}$

— the same object

— the corresponding representation theoretic informations are related by level-rank duality

I need to go back the starting point
to explain the relation to the level-rank duality.

$\mathfrak{M}_0^{reg}(\vec{v},\vec{w})$: framed moduli sp of $GL(r)$-b'dles
on $\mathbb{C}^2/\Gamma = \mathbb{C}^2/\mathbb{Z}_{n+1}$

framing = fiber at ∞ : $\Gamma \to GL(r)$
$= \sum w_i \rho_i$

$\vec{v}$: $H^1(E(-\ell_\infty)) = \sum_i v_i \rho_i$

- quiver var : we understand these as pairs of affine dominant wts of $\widehat{\mathfrak{sl}}_n$ by

$\vec{w} = \sum w_i \Lambda_i$, $\vec{v} = \sum v_i \alpha_i$

$\mathfrak{M}_0^{reg}(\vec{v},\vec{w}) \neq \phi \Rightarrow \vec{w}-\vec{v}$: dominant $\leq \vec{w}$
rep. of Γ at 0

- Gr

$\mathfrak{M}_0^{reg}(\vec{v},\vec{w}) = M_\mu^\lambda$: "transversal slice" of "Gr^μ" in "$\overline{Gr^\lambda}$"

λ, μ : dominant wts of $\widehat{\mathfrak{sl}}_r$ of level n

by $\boxed{\mu = {}^t\vec{w} \quad , \lambda = {}^t(\vec{w}-\vec{v})}$ transposed Young diagrams

① IC sheaves of strata

$H^*(IC(\overline{\mathfrak{M}_0^{reg}(\vec{v},\vec{w})}))$ Uhlenbeck partial cptification

$\underset{gr}{=}$ weight space $V_{\widehat{sl}(r)}(\lambda)_\mu$ t : transpose

$\underset{quiver}{=}$ multiplicity $[V_{\widehat{sl}(n)}(\vec{w}^t-\vec{v}^t) : \operatorname{Res} M(\vec{w})|_{q=1}]$

std module : representation of $U_q(L\widehat{sl}(n))$
toroidal alg.

$\operatorname{Res} : L\widehat{sl}(n)\text{-mod} \to \widehat{sl}(n)\text{-mod}$

For type $\hat{A}$: $\operatorname{Res} M(\vec{w})$ = tensor prod. of level 1's of $\widehat{gl}(n)$

In level-rank duality
weight multiplicity$_{\widehat{sl}(r)}$ = tensor prod. decomp.$_{\widehat{gl}(n)}$ of level 1's

Rem. Both have interpretations of q-analogs, but = are not checked in q-analog.
(q-analogs : deg of H^*)

② MV cycles v.s. tensor product varieties

MV cycles are certain varieties in Gr giving a basis of wt spaces.

It has a structure of the Kashiwara crystal.

original tensor product varieties

...... lagrangian subvarieties in $\mathfrak{M}(\vec{v}, \vec{w})$ (resolution)

· the set of irreducible components has the structure of tensor product crystal

$$\begin{array}{ccc} \mathfrak{M}(\vec{v}, \vec{w}) & \longrightarrow & \overline{\mathfrak{M}_0^{reg}(v, w)} \\ \cup & & \cup \\ \tilde{\mathfrak{Z}} & \longrightarrow & \text{image} \end{array}$$

an irreducible component survives

$\Leftrightarrow$ it corresponds to highest wt vectors

Therefore irreducible components of the image gives a basis of Tensor prod. multiplicity space.

We need a slight modification because we need Tensor product of $\widehat{gl}(n)$-rep's instead of $\widehat{sl}(n)$

③ convolution diagram

→ tensor prod in Gr and branching in quiver

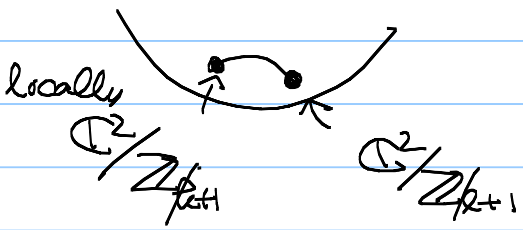
double affine analog of $\quad \mathrm{Gr} \tilde{\times} \mathrm{Gr} \xrightarrow{\varpi} \mathrm{Gr}$

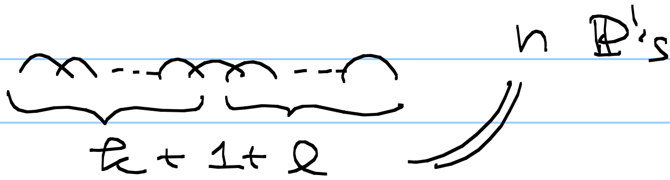
$$\varpi_*(IC_\lambda \tilde{\boxtimes} IC_\mu) = \bigoplus m^{\nu}_{\lambda\mu} IC_\nu$$

[BF] proposal :

$\mathbb{C}^2/\mathbb{Z}_{n+1} \longleftarrow X'$: partial resolution

$(\nwarrow \; X$: minimal resolution $)$





$$\overline{\text{moduli of instantons on } X'} \xrightarrow{\varpi} \overline{\text{moduli of inst. on } \mathbb{C}^2/\mathbb{Z}_{n+1}}$$

● ϖ makes sense in the quiver picture and its representation theoretic meaning is the branching $(\mathfrak{sl}(k+1)\times \mathfrak{sl}(l+1))^{\wedge} \hookrightarrow \widehat{\mathfrak{sl}}(n+1)$

(central extension is in common)

bundles on X' are parametrised again by $\vec{v}, \vec{w}$

but $\mathfrak{M}'(\vec{v}, \vec{w}) \neq \phi \Leftrightarrow \vec{w} - \vec{v}$: dominant as

{ $(\mathfrak{sl}(k+1) \times \mathfrak{sl}(\ell+1))^{\wedge}$ - wt

rep of $\mathbb{Z}/_{k+1}$ & $\mathbb{Z}/_{\ell+1}$

Th. $\varpi_*(IC(\overline{\mathfrak{M}'(\vec{v}, \vec{w})})) = \bigoplus_{\vec{v}'} \underbrace{a_{\vec{v}\vec{v}'}}_{\shortparallel} \cdot IC(\overline{\mathfrak{M}_0^{reg}(\vec{v}', \vec{w})})$

$[V_{(\mathfrak{sl}(k+1) + \mathfrak{sl}(\ell+1))^{\wedge}}(\vec{w} - \vec{v}) : \mathrm{Res}(V_{\widehat{\mathfrak{sl}(n)}}(\vec{w} - \vec{v}'))]$

Rem ϖ : semismall

$\Longrightarrow a_{vv'}$: integers , no q-analog.

This is an application of a general theory of partial resolutions & restrictions in quiver var.

$\mathfrak{M}_0, \mathfrak{M}'$ or $\mathfrak{M}$

∵ sheaves on X

Γ-equiv sheaves on $\mathbb{C}^2$

are all GIT quotients of the common affine variety by various choices of stability conditions

The space of stability conditions

$\cong \mathfrak{h}_{\mathbb{Q}}$ (Cartan subalg.)

Chamber structure by root hyperplanes

param $\zeta \in \bigcap$ hyperplanes

$$\rightsquigarrow \mathfrak{g}^{\zeta} = \mathfrak{h} \oplus \bigoplus_{\langle \zeta, \alpha \rangle = 0} \mathfrak{g}_{\alpha} \subset \mathfrak{g} \quad \text{subalgebra}$$

$\mathfrak{M}_{\zeta} \longrightarrow \mathfrak{M}_0$ controls the branching $\mathfrak{g} \downarrow \mathfrak{g}^{\zeta}$

ex1. $I \supset I^0$ subset

$$\zeta : \quad \langle \zeta, \alpha_i \rangle = 0 \quad i \in I^0$$
$$\langle \zeta, \alpha_i \rangle > 0 \quad i \in I \setminus I^0$$

$\Rightarrow \mathfrak{g}^{\zeta} = \mathfrak{g}_{I^0}$: Levi part of the parabolic ($\oplus$ Cartan)

ex2. I : affine $= I_0 \cup \{0\}$

$$I_0 \cup I_0^0$$

$$\zeta : \quad \langle \zeta, \delta \rangle = 0$$
$$\langle \zeta, \alpha_i \rangle = 0 \quad i \in I_0^0$$
$$\langle \zeta, \alpha_i \rangle > 0 \quad i \in I_0 \setminus I_0^0$$

$$\Rightarrow \mathfrak{g}^{\zeta} = \widehat{\mathfrak{g}_{I_0^0}}$$

The principle gives a further conjecture:

④ quantum toroidal action
in quiver var. v.s. ? in $\mathcal{G}r$?
⋮
IC(stratum in $\mathfrak{M}_0^{reg}(\vec{v},\vec{w})^{\mathbb{C}^*}$)

$\mathbb{C}^*$ $(x,y) \mapsto (tx, ty)$
& change of the framing
$\mathbb{C}^* \to GL(r)$ commuting with $\Gamma \xrightarrow{\rho_\infty} GL(r)$

The $\mathbb{C}^*$-fixed point set makes sense in $\mathcal{G}r$.
But what is its representation theoretic meaning?

I do <u>not</u> know even for the usual affine Grassmann.